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BRST theory without hamiltonian and lagrangian

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Abstract: We consider a generic gauge system, whose physical degrees of freedom are obtained by restriction on a constraint surface followed by factorization with respect to the action of gauge transformations; in so doing, no hamiltonian structure or action principle is supposed to exist. For such a generic gauge system we construct a consistent BRST formulation, which includes the conventional BV lagrangian and BFV hamiltonian schemes as particular cases. If the original manifold carries a weak Poisson structure (a bivector field giving rise to a Poisson bracket on the space of physical observables) the generic gauge system is shown to admit deformation quantization by means of the Kontsevich formality theorem. A sigma-model interpretation of this quantization algorithm is briefly discussed.

Keywords: BRST Quantization, Gauge Symmetry, BRST Symmetry.

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Contents

1. Introduction

The standard BRST theory of gauge systems [1] starts from either lagrangian formulation of the original dynamics (the BV method [2]) or hamiltonian constrained formalism (the BFV method [3]). This paper is about constructing a BRST embedding for the gauge systems, whose physical degrees of freedom are defined merely by restriction to a smooth submanifold (called shell) accompanied by factorization with respect to the action of an on-shell integrable vector distribution (gauge algebra generators). In this quite general setting we start with, the shell can be thought of, for example, as defined either by equations of motion (not necessarily lagrangian) or by a constraint surface (not necessarily hamiltonian). Accordingly, the gauge algebra generators, foliating the shell into the gauge orbits, are required neither to annihilate some action functional nor to be induced by a set of hamiltonian first-class constraints.

In most cases of gauge systems the lagrangians are known, though the absence of an action principle is not uncommon: the equations for interacting higher-spin gauge fields [4], for example, do not admit the action principle, at least in their present form. In the recent paper [5], it is remarked that any given BRST differential in the space of trajectories allows to identify equations of motion and gauge symmetries independently of the question of the existence of a lagrangian for these equations. Using this observation, a general BRST formulation for Vasiliev's method of unfolding [4] was given in [5] for the case of free field theories associated to first-quantized constrained systems.

In the present paper, proceeding from the understanding that the physical observables are the functions defined modulo on-shell vanishing terms and remained constant along the gauge orbits, we formulate a generic gauge theory with no reference to any lagrangian or hamiltonian structure. As a next step we suggest a uniform scheme for the BRST embedding of the generic gauge system whose dynamics is defined just by equations of motion and constraints, with no Poisson bracket or action principle involved. This more general view may also have some impact on understanding of the gauge systems which can admit a conventional lagrangian or hamiltonian description as well. In particular, in such a general setting, the gauge generators can be much less tightly bound to the shell, than it is usually required in constrained hamiltonian dynamics. Being applied to hamiltonian systems, this formalism allows for factorization of mixed-class constrained surfaces by any on-shell integrable transformations, not only by the hamiltonian action of the first class constraints, as it is usually implied in the conventional BRST-BFV scheme. Some approaches of this sort to the mixed class systems have been discussed, for instance, in ref. [6].

In order to quantize the generic gauge system, we study a possibility to endow only the space of physical observables with a Poisson algebra structure; in so doing, no Poisson structure is supposed to exist on the original space of the system or on its BRST extension. The existence of the Poisson brackets on the entire space and the off-shell compatibility of the Poisson structure with the gauge algebra seem to be excessive requirements from the viewpoint of a consistent physical interpretation and deformation quantization of gauge systems.

Geometrically, the usual requirements of BFV method for the Poisson brackets to possess the off-shell Jacobi identity and the gauge generators to be off-shell hamiltonian vector fields seem as superficial as demanding the gauge generators to form an off-shell integrable distribution. The later requirement has been recognized as an excessive restriction long ago. In refs. [7, 8] weak Poisson brackets, satisfying the Jacobi identity only on shell, have been already studied in the BRST framework. The relevance of the weak Poisson brackets to the covariant quantization of superstrings was discussed [9]. In this work we find that the concept of the weak Poisson brackets can be further relaxed for the gauge systems: it is sufficient to have the on-shell Jacobi identity only for physical observables (i.e. gauge invariant values), not for arbitrary functions.

We show that such a generalized notion of the weak Poisson structure fits well to the general concept of gauge systems proposed in ref. [10]. More precisely, it is possible to generate all the structure relations underlying both the gauge symmetry and the weak Poisson structure from a single master equation $(S, S) = 0$, where S is an even ghost-number-2 function on an appropriately chosen anti-Poisson manifold. The physical dynamics is specified by means of an odd, ghost-number-1 function H subject to the equation $(S, H) = 0$. Given the generic gauge system on the original manifold M, the functions S and H are systematically constructed by solving the master equations on the properly ghost extended M.

The functions S and H serve as prerequisites to perform the deformation quantization of the generic gauge system. The quantum counterpart of the weak Poisson structure is a weakly associative $*$-product. The latter can be constructed by applying the Kontsevich formality theorem. The "weakness" means that upon the quantum reduction the $*$-product induces an associative quantum multiplication for physical observables. It turns out that the direct application of the formality map to S and H may face the problem of quantum anomalies (i.e. the values which break the existence of quantum reduction) and we comment on how to remove these anomalies by a proper redefinition of S and H.

Finally, we show that, as for usual Poisson manifolds, the deformation quantization of the weak Poisson structures can be explored in terms of a 2-dimensional topological sigma-model associated to S. This approach can be used as the basis for definition and (non-rigor) quantization of the generic gauge systems associated with non-canonical antibrackets $(\cdot\,,\cdot)$. The non-canonical antibrackets become inevitable, e.g., when the normal bundle to the shell and/or the tangent bundle of the on-shell gauge foliation admit no flat connection.

2. Generic constraints and gauge algebra

In any gauge theory, true physical degrees of freedom appear as a result of reduction of the original configuration or phase-space manifold. Two different ways are known for the reduction: i) a direct restriction of the dynamics to a submanifold (which we call a shell), and ii) factorization with respect to the action of vector fields generating the gauge invariance on shell. The shell can be understood, for example, as a constraint surface in constrained hamiltonian dynamics or as a mass-shell of lagrangian gauge theory. The compatibility conditions between restriction and factorization requires the gauge generators to define an on-shell integrable distribution. In other words, the generic gauge transformations are not necessarily integrable in the entire configuration/phase space, foliating only the shell into the gauge orbits. The physical observables are defined modulo on-shell vanishing terms and are required to be constant along the gauge orbits. In this section, we describe the algebra of the physical observables of the generic gauge system where the physical dynamics emerges from the reduction of base manifold to any shell (not necessarily hamiltonian constraint surface or lagrangian mass-shell) followed by factorization with respect to any on-shell integrable distribution.

Let M be a smooth manifold with local coordinates x^i. Consider a nonempty subset $\Sigma \subset M$, called the shell, which is determined by a system of m equations

$$T_a(x) = 0\,, \qquad a = 1, \dots, m\,, \tag{2.1}$$

called the constraints. The standard regularity condition

$$\mathrm{rank}\left(\frac{\partial T_a}{\partial x^i}\right)\Big|_\Sigma = m \tag{2.2}$$

ensures that Σ is a smooth submanifold of $\mathrm{codim}\Sigma = m$. In the language of constrained dynamics, this means that the constraints $T_a(x)$ are supposed to be irreducible:

$$Z^a(x)T_a(x) = 0 \;\Rightarrow\; \exists\, A^{ab}(x) = -A^{ba}(x) : \, Z^a = A^{ab}T_b\,. \tag{2.3}$$

If desired, Σ can be regarded as a mass-shell of a lagrangian gauge system (or the shell of the equations of motion of a non-lagrangian theory), though this requires constraints T_a to be reducible. In other words, the on-shell nonvanishing null-vectors Z^a have to exist, defining the Noether identities between the equations of motion[1] $T_a = 0$. It is not a problem to

[1]The generators of identities do not necessarily coincide with the generators of gauge symmetries for a non-lagrangian gauge system.

extend the consideration to the reducible case (this will be done in the next work), but in this paper we confine ourselves with the irreducible case for the sake of technical simplicity. So it is slightly easier to think of T_a as constraints (e.g. the hamiltonian ones).

Consider a set of n vector fields $R_a = R^i_\alpha \partial_i$ on M, which are tangent to Σ, i.e.

$$[R_\alpha, T_a] \equiv R^i_\alpha \partial_i T_a = E^b_{\alpha a} T_b \,, \tag{2.4}$$

for some $E^c_{\alpha b}(x) \in C^\infty(M)$. Hereafter the square brackets $[\,\cdot\,,\,\cdot\,]$ stand for the Schouten commutator in the exterior algebra $\Lambda(M) = \bigoplus \Lambda^k(M)$ of smooth polyvector fields on M. [2] The vector fields R_α are assumed to be linearly independent on Σ and to form an on-shell integrable distribution. In other words,

$$\mathrm{rank}(R^i_\alpha)|_\Sigma = n \tag{2.5}$$

and

$$\frac{1}{2}[R_\alpha, R_\beta] = U^\gamma_{\alpha\beta} R_\gamma + T_a A^a_{\alpha\beta} \,, \tag{2.6}$$

for some vector fields $A^a_{\alpha\beta} \in \Lambda^1(M)$. The vector fields R_α will be called gauge generators if the conditions (2.4), (2.6) are satisfied. The irreducibility condition (2.5) can be relaxed, but in this paper we consider only this simplest case.

Usually the gauge generators are assumed to be either hamiltonian vector fields for the first class constraints or the annihilators of the action functional. In these standard cases, the relations (2.4), (2.6) hold true and, as is argued below, it is the condition which defines the gauge system even when no hamiltonian or lagrangian structure is admissible on the original manifold. Since, in general, the integrability of the vector distribution $\{R_\alpha\}$ takes place only on the submanifold Σ, determined by the system of constraints T_a, we will refer to the involution relations (2.4), (2.6) as a *constrained gauge algebra*. On shell, the constrained gauge algebra is nothing but the Lie algebroid, off shell it is not, in general.

Let $\mathcal{F}(\Sigma)$ denote the foliation associated to the integrable distribution $\{\bar{R}_\alpha = R_\alpha|_\Sigma\}$ and let N be the corresponding space of leaves (the points of N are just leaves of the foliation $\mathcal{F}(\Sigma)$). In general, N may not be a smooth, Hausdorff manifold. Nonetheless, we can define the space $\mathcal{T}(N)$ of "smooth" tensor fields on N as the space of invariant tensor fields on Σ:

$$L_{\bar{R}_\alpha} S = 0 \;\Leftrightarrow\; S \in \mathcal{T}(\Sigma)^{\mathrm{inv}} \,. \tag{2.7}$$

In what follows we deal mainly with the algebra $\Lambda(N) = \Lambda(\Sigma)^{\mathrm{inv}}$ of smooth polyvector fields on N. In view of the regularity conditions (2.2) and (2.5), the space $\Lambda(N)$ can also be identified with equivalence classes of projectible polyvector fields on M: Two k-vectors $A, B \in \Lambda^k(M)$ are said to be *equivalent* ($A \sim B$) iff

$$A - B = T_a K^a + S^\alpha \wedge R_\alpha \,, \tag{2.8}$$

for some $K^a \in \Lambda^k(M)$, $S^\alpha \in \Lambda^{k-1}(M)$, and the k-vector A is said to be *projectible* (onto N) if

$$[T_a, A] \sim 0 \,, \qquad [R_\alpha, A] \sim 0 \,. \tag{2.9}$$

[2]The Schouten commutator is the unique bracket of polyvector fields which agrees with the Lie bracket of vector fields and is a graded derivation of degree -1 of the exterior product.

The polyvectors of the form $T_a K^a + S^\alpha \wedge R_\alpha$ will be called *trivial*. These form a linear subspace $\Lambda(M)^{\rm triv} = \{A \in \Lambda(M) | A \sim 0\}$. Since the trivial polyvectors are obviously projectible, we can write

$$\Lambda(M)^{\rm triv} \subset \Lambda(M)^{\rm pr} \subset \Lambda(M)\,, \tag{2.10}$$

and set

$$\Lambda(N) = \Lambda(M)^{\rm pr}/\Lambda(M)^{\rm triv}\,. \tag{2.11}$$

It is the space $\Lambda^0(N) = \Lambda^0(M)^{\rm pr}/\Lambda^0(M)^{\rm triv}$, which is regarded as the *space of physical observables* of the generic gauge system. In the less formal physical terminology, the physical observables are the gauge-invariant functions taken modulo on-shell vanishing terms. Note that this definition does not refer to any action functional or hamiltonian structure.

Another relevant space is $\Lambda^1(N) = \Lambda^1(M)^{\rm pr}/\Lambda^1(M)^{\rm triv}$. Classically, the physical dynamics on N is specified by a projectible vector field $V \in \Lambda^1(M)^{\rm pr}$: The evolution of a physical observable represented by $O \in C^\infty(M)^{\rm pr} \equiv \Lambda^0(M)^{\rm pr}$ is governed by the equation

$$\dot{O} = [V, O]\,, \tag{2.12}$$

where the overdot stands for the time derivative. If O is a projectible (trivial) function at the initial time moment, it will remain a projectible (trivial) function in future as long as V is a projectible vector field.

Bearing in mind the deformation quantization of the classical gauge system (2.12), we have to equip the commutative algebra of functions $C^\infty(N) \equiv \Lambda^0(N)$ with a Poisson structure. For this end, we introduce a projectible bivector field $P \in \Lambda^2(M)^{\rm pr}$ subject to the weak Jacobi identity

$$[P, P] \sim 0\,, \tag{2.13}$$

and set

$$\{A, B\} = P^{ij}\partial_i A \partial_j B\,, \quad \forall A, B \in C^\infty(M)\,. \tag{2.14}$$

Then for all projectible functions $O_1, O_2, O_3 \in C^\infty(M)^{\rm pr}$ and a trivial function $A \sim 0$ we have

1. $\{O_1, A\} \sim 0$,
2. $\{O_1, O_2\} \in C^\infty(M)^{\rm pr}$,
3. $\{\{O_1, O_2\}, O_3\} + \text{c.p.}(O_1, O_2, O_3) \sim 0$.

Relations 1), 2) are just reformulations of the projectibility conditions for P, whereas 3) is equivalent to the weak Jacobi identity (2.13). Together relations 1),2),3) mean that the brackets (2.14) define the Poisson algebra structure on the space of physical observables. A projectible bivector P, satisfying the above relations, will be called a *weak Poisson structure* associated to the constrained gauge algebra (2.4), (2.6).

A projectible vector field V defines the Poisson vector field on N if and only if

$$[V, P] \sim 0\,. \tag{2.15}$$

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In what follows we will refer to V as a *weak Poisson vector field*. The condition (2.15) will be crucial upon formulating a quantum counterpart of the classical equation of motion (2.12).

Applying the Jacobi identity to the Schouten brackets (2.4), (2.6), (2.13) and taking into account the regularity conditions (2.2), (2.5), one can derive an infinite set of higher structure relations underlying the constrained gauge algebra endowed with a weak Poisson structure. In the next section we present a systematic BRST-like procedure for generating all these relations. Also this BRST formalism will serve as basis for the deformation quantization of the generic gauge system by means of the formality theorem (see section 4).

3. BRST embedding of a generic gauge system

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The BRST embedding of any gauge system always starts with an appropriate extension of the base manifold M with ghost variables. The conventional ways to introduce ghosts are different for the hamiltonian and lagrangian BRST schemes. In this section we suggest a uniform scheme for the BRST embedding of the generic gauge system which does not involve Poisson structure or action. Here we consider the constraints (2.1) and gauge generators (2.4), (2.6) as scalar functions and vector fields, so the bundle of ghost variables is trivial. The generalization to nontrivial vector bundles will be given in section 5.

Consider a supermanifold $\mathcal{M}$ associated with the trivial vector bundle $M \times \Pi\mathbb{R}^m \times \Pi\mathbb{R}^n \to M$. Let η_a and c^α denote linear coordinates on the odd vector spaces $\Pi\mathbb{R}^m$ and $\Pi\mathbb{R}^n$, respectively. If one regards the shell $\Sigma : T_a = 0$ as the surface of hamiltonian first class constraints, identifying the gauge generators R_α with the hamiltonian vector fields $R_a = \{T_a, \cdot\}$, then c^α and η_a will be the standard BFV ghosts and ghost momenta respectively[3] (though in our approach they are no longer assumed to be conjugated variables w.r.t. any Poisson bracket). To construct the classical BRST embedding of the generic gauge it would be sufficient, in principle, to work only with the supermanifold $\mathcal{M}$, if the presence of a weak Poisson structure was not crucial (see comments after rel. (3.25)). To trace the compatibility of the BRST embedding with the Poisson algebra of physical observables we need, however, the further extension of $\mathcal{M}$ to its odd cotangent bundle $\Pi T^*\mathcal{M}$. There is an obvious isomorphism

$$\Pi T^*\mathcal{M} \sim (\mathbb{R}^m)^* \times (\mathbb{R}^n)^* \times \Pi\mathbb{R}^m \times \Pi\mathbb{R}^n \times \Pi T^* M\,, \tag{3.1}$$

where the asterisk means passing to a dual vector space. Let $\overset{*}{\eta}{}^a$, $\overset{*}{c}_\alpha$ and $\overset{*}{x}_i$ denote linear coordinates on $(\mathbb{R}^m)^*$, $(\mathbb{R}^n)^*$ and $T^*_x M$, respectively.

[3] For a lagrangian gauge system, the constraints $T_a = 0$ have to be understood as equations of motion. The vector fields R's are then regarded as generators of the gauge transformations, c's are naturally identified with the standard BV ghosts, and η's play the role of BV antifields. Note that in the presence of gauge symmetries the lagrangian equations of motion $T_a = 0$ are necessarily reducible. In the spirit of usual ideas of the BRST theory, the ghost for ghost are to be introduced in this case. Here, we deal only with irreducible case, although the generalization to reducible constraints (which is necessary to describe accurately a lagrangian gauge theory) is a technical problem which will be considered elsewhere.

	x	$\overset{*}{x}$	c	$\overset{*}{c}$	η	$\overset{*}{\eta}$
ϵ=Grassman parity	0	1	1	0	1	0
gh = ghost number	0	1	1	0	-1	2
Deg = polyvector degree	0	1	0	1	0	1
deg = resolution degree	0	0	0	1	1	0

Table 1: The degrees of local coordinates on $\Pi T^*\mathcal{M}$.

The supermanifold $\Pi T^*\mathcal{M}$ can be endowed with various $\mathbb{Z}$-gradings in addition to the basic $\mathbb{Z}_2$-grading (the Grassman parity). As usual, this is achieved by prescribing certain integer degrees to the local coordinates, see table 1. For brevity, the polyvector and resolution degrees will be referred to as p- and r-degrees respectively. Splitting the coordinates into *fields* $\phi^A = (x^i, c^\alpha, \eta_a)$ and *antifields* $\phi^*_A = (\overset{*}{x}_i, \overset{*}{c}_\alpha, \overset{*}{\eta}^a)$, we can write

$$\mathrm{gh}(\phi^*_A) = -\mathrm{gh}(\phi_A) + 1\,, \qquad \epsilon(\phi^*_A) = \epsilon(\phi^A) + 1\,, \qquad \mathrm{Deg}(\phi^*_A) = 1\,, \qquad \mathrm{Deg}(\phi^A) = 0\,. \tag{3.2}$$

Here the coordinates $\overset{*}{c}_\alpha$ are treated as formal variables although they have the same parity and the ghost number as the local coordinates x^i. The distinctions between x's and $\overset{*}{c}$'s are indicated by the auxiliary p- and r-degrees.

The odd cotangent bundle $\Pi T^*\mathcal{M}$ carries the canonical antisymplectic structure. The corresponding anti-Poisson brackets of fields and antifields read

$$(\phi^*_A, \phi^B) = \delta^B_A\,, \qquad (\phi^*_A, \phi^*_B) = 0\,, \qquad (\phi^A, \phi^B) = 0\,. \tag{3.3}$$

Now we introduce a pair of master equations generating all the structure relations underlying the constrained gauge algebra (2.1)–(2.6) equipped with a weak Poisson structure P and a weak Poisson vector field V. The equations read

$$(S, S) = 0\,, \qquad (S, H) = 0\,, \tag{3.4}$$

where the functions S and H are subject to the following grading and regularity conditions:

$$\mathrm{gh}(S) = 2\,, \qquad \epsilon(S) = 0\,, \qquad \mathrm{Deg}(S) > 0\,, \tag{3.5}$$

$$\mathrm{gh}(H) = 1\,, \qquad \epsilon(H) = 1\,, \qquad \mathrm{Deg}(H) > 0\,, \tag{3.6}$$

$$\mathrm{rank}\left(\frac{\partial^2 S}{\partial \phi^A \partial \phi^*_B}\right)\Bigg|_{dS=0} = (n, m)\,. \tag{3.7}$$

Note that the last inequalities in (3.5) and (3.6) are equivalent to vanishing of S and H on the lagrangian surface $\mathcal{M} \subset \Pi T^*\mathcal{M}\,:\ \phi^*_A = 0$.

The general expansion for S, compatible with the grading (3.5), reads

$$\begin{aligned} S =\ & \overset{*}{\eta}^a T_a + \overset{*}{x}_i R^i_\alpha c^\alpha + \overset{*}{x}_j \overset{*}{x}_i P^{ij} + \overset{*}{c}_\gamma \left(U^\gamma_{\alpha\beta} c^\beta c^\alpha + \overset{*}{x}_j \overset{*}{x}_i V^{\gamma ij} + \overset{*}{x}_i c^\alpha W^{\gamma i}_\alpha + \overset{*}{\eta}^a Y^\gamma_a \right) + \\ & + \eta_a \left(\overset{*}{x}_i A^{ai}_{\alpha\beta} c^\beta c^\alpha + \overset{*}{x}_j \overset{*}{x}_i B^{aij}_\alpha c^\alpha + \overset{*}{x}_k \overset{*}{x}_j \overset{*}{x}_i D^{aijk} + \overset{*}{\eta}^b E^a_{b\alpha} c^\alpha + \overset{*}{x}_i F^{ai}_b \overset{*}{\eta}^b \right) + \cdots\,, \end{aligned} \tag{3.8}$$

where dots stand for the terms of r-degree ≥ 2. The coefficients at the first three terms (all of r-degree 0) are identified respectively with the constraints (2.1), the gauge algebra generators (2.4), (2.6) and the weak Poisson bivector (2.13). Upon this identification the regularity conditions (2.2), (2.5) are expressed by rel. (3.7).

Up to the first order in r-degree a general expansion for H is given by

$$H = V^i \overset{*}{x}_i + \overset{*}{c}_\alpha \left(W^\alpha_\beta c^\beta + \overset{*}{x}_i G^{\alpha i} \right) + \eta_a \left(M^a_b \overset{*}{\eta}{}^b + \overset{*}{x}_i \overset{*}{x}_j L^{aji} + \overset{*}{x}_i N^{ai}_\alpha c^\alpha \right) + \cdots , \tag{3.9}$$

where the first term is identified with the projectible vector field entering the equation of motion (2.12).

The existence of solutions to the master equations (3.4) is proved by the standard tools of homological perturbation theory [1, 11]. Below, we sketch the proof for S; for H, the existence theorem is proved in a similar way.

Substituting the general expansion

$$S = \sum_{n=0}^{\infty} S_n , \qquad \deg(S_n) = n , \tag{3.10}$$

in the first equation of (3.4), we arrive at a chain of equations of the form

$$\delta S_{n+1} = K_n(S_0, \ldots, S_n) , \qquad \deg(K_n) = n , \tag{3.11}$$

where

$$\delta = T_a \frac{\partial}{\partial \eta_a} + \overset{*}{x}_i R^i_\alpha \frac{\partial}{\partial \overset{*}{c}_\alpha} \tag{3.12}$$

is a nilpotent operator decreasing the r-degree by one unit,

$$\delta^2 = 0 , \qquad \deg(\delta) = -1 , \tag{3.13}$$

and K_n involves the antibrackets of the S's of lower order. Let $\mathcal{H}(\delta) = \bigoplus \mathcal{H}_n(\delta)$ denote the corresponding cohomology group graded naturally by r-degree. ¿From the regularity conditions (2.2), (2.5) it follows that

$$\mathcal{H}_n(\delta) = 0 \quad \text{for} \quad n > 0 . \tag{3.14}$$

Expanding the first equation $\delta S_1 = K_0(S_0)$ in the ghost variables one recovers the involution relations (2.4), (2.6), the projectibility conditions (2.9) for the weak Poisson bivector P,

$$[T_a, P] = -Y^\alpha_a \wedge R_\alpha - T_b F^b_a , \quad [R_\alpha, P] = W^\beta_\alpha \wedge R_\beta - T_a B^a_\alpha , \tag{3.15}$$

and the weak Jacobi identity (2.13),

$$\frac{1}{2}[P, P] = -V^\alpha \wedge R_\alpha - T^a D_a . \tag{3.16}$$

Now the existence of n-order solution can be proved by induction. Indeed, from the Jacobi identity $(S, (S, S)) \equiv 0$ it follows that the r.h.s. of n-th equation is δ-closed, provided that the previous $(n-1)$-th equations are satisfied. In view of (3.14), this δ-closed expression is exact, i.e. some S_{n+1} exists obeying (3.11). Using the induction on n one can also see that $S_{n+1}|_{\phi^*=0} = 0$, as the operator δ does not change p-degree, while the antibrackets decrease the degree by one.

Thus, the master equation $(S, S) = 0$ is proved to be soluble for an arbitrary constrained gauge algebra and a weak Poisson structure.

Notice that neither S nor H are uniquely determined by the master equations (3.4). The reason is that one can always make a canonical transformation $f : \Pi T^*\mathcal{M} \to \Pi T^*\mathcal{M}$ (i.e. a diffeomorphism respecting the canonical antibrackets (3.3)) and get new solutions $S' = f^*(S)$ and $H' = f^*(H)$ to the same equations. Consider, for instance, a finite anti-canonical transformation

$$F' = e^{(G,\cdot)}F = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(G, (G, \ldots (G}_{n}, F) \ldots), \qquad \forall F \in C^\infty(\Pi T^*\mathcal{M}) . \tag{3.17}$$

generated by the odd, ghost number 1 function

$$G = c^\alpha A_\alpha^\beta(x) \overset{*}{c}_\beta + \eta_a B_b^a(x) \overset{*}{\eta}{}^b + \eta_a c^\alpha Z_\alpha^{ai}(x) \overset{*}{x}_i . \tag{3.18}$$

By applying this transformation to S and H, one redefines the structure functions entering the expansions (3.8) and (3.9), including those ones which are identified with the original constraints and gauge generators:

$$T_a \to T'_a = \tilde{A}_a^b T_b , \qquad R_\alpha \to R'_\alpha = \tilde{B}_\alpha^\beta \left(R_\beta + T_a \tilde{Z}_\beta^a \right) , \tag{3.19}$$

where

$$\tilde{A} = e^A , \qquad \tilde{B} = e^B , \qquad \tilde{Z}^i = \int_0^1 e^{tA} Z^i e^{-tB} dt . \tag{3.20}$$

Relations (3.19) reflect an inherent ambiguity in the definition of the constraints and gauge algebra generators. The invariant geometrical meaning can be assigned only to the shell Σ (a particular choice of the constraints T_a is not significant if they have the same zero locus), and the gauge foliation $\mathcal{F}(\Sigma)$ (various choices of the vector fields R_α are all equivalent if they define the same on-shell integrable distribution). By adding to G the term $W^i(x) \overset{*}{x}_i$, $W^i\partial_i$ being a complete vector field on M, one generates just a diffeomorphism of the original manifold M. The other possible terms that one may add to G do not contribute to the transformations (3.19). A deeper geometric insight into the nature of these transformations will be given in section 5, where the T's and R's are considered to be sections of appropriate vector bundles with connection rather than sets of functions and vector fields.

Consider another interpretation for the master equations (3.4) in terms of polyvector algebra. It is based on an obvious identification of the odd Poisson algebra on $\Pi T^*\mathcal{M}$ with the exterior algebra of polyvector fields $\Lambda(\mathcal{M}) = \bigoplus \Lambda^n(\mathcal{M})$ on $\mathcal{M}$ endowed with the Schouten brackets. Upon this interpretation the anti-fields ϕ_A^* play the role of the natural frame ∂_A in the fibers of the odd tangent bundle $\Pi T\mathcal{M}$ and the rank of a polyvector coincides with the p-degree of the corresponding superfunction. Each homogeneous subspace $\Lambda^n(\mathcal{M}) = \bigoplus \Lambda_m^n(\mathcal{M})$ is further graded by the ghost number $m = 0, 1, \ldots$.

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The expansions of S and H w.r.t. the p-degree read

$$S = \sum_{n=1}^{\infty} S^n = Q^A \phi^*_A + \Pi^{AB}\phi^*_B\phi^*_A + \Psi^{ABC}\phi^*_C\phi^*_B\phi^*_A + \cdots\,, \tag{3.21}$$

$$H = \sum_{n=1}^{\infty} H^n = \Gamma^A \phi^*_A + \Xi^{AB}\phi^*_A\phi^*_B + \cdots\,, \tag{3.22}$$

where $\mathrm{Deg}(S^n) = \mathrm{Deg}(H^n) = n$. Note that the contributions of 0-vectors to S and H are prohibited by (3.5), (3.6).

To simplify notation, we use the same letter F for a superfunction $F(\phi^A, \phi^*_B) \in C^\infty(\Pi T^*\mathcal{M})$ and for the corresponding polyvector field $F(\phi^A, \partial_B) \in \Lambda(\mathcal{M})$. Then the master equations on S and H yield two sequences of relations for the homogeneous polyvectors:

$$[S,S] = 0 \;\Rightarrow\; [Q,Q] = 0\,, \qquad [Q,\Pi] = 0\,, \qquad [\Pi,\Pi] = -2[Q,\Psi]\,, \quad \cdots\,, \tag{3.23}$$

$$[S,H] = 0 \;\Rightarrow\; [Q,\Gamma] = 0\,, \qquad [\Gamma,\Pi] = [Q,\Xi]\,, \quad \cdots\,. \tag{3.24}$$

The first relation in (3.23) means that

$$Q = T_a \frac{\partial}{\partial \eta_a} + c^\alpha R^i_\alpha \frac{\partial}{\partial x^i} + c^\beta c^\alpha U^\gamma_{\alpha\beta}\frac{\partial}{\partial c^\gamma} + \eta_a A^{ai}_{\alpha\beta} c^\beta c^\alpha \frac{\partial}{\partial x^i} + \cdots \tag{3.25}$$

is an integrable odd vector field on $\mathcal{M}$ of ghost number 1. In mathematics, such a value is known as a homological vector field (see e.g. [12]). From the physical viewpoint, it is Q which serves as a BRST operator, generating classical BRST transformations on $\mathcal{M}$ (i.e. on the original manifold M extended with the ghosts c's and η's). To construct Q, one could directly solve the nilpotency condition $[Q,Q] = 0$ without resorting to antifields ϕ^*_A. Obviously, Q contains all the information about the constrained gauge algebra (2.1), (2.4), (2.6) defined independently of the weak Poisson structure P. The classical evolution (2.12) is generically described by the weak Poisson vector field V. If one identifies the base manifold M with the space of trajectories of a gauge system, the constraints $T_a = 0$ can be interpreted as equations of motion; in so doing, the vector field V can be regarded as the generator of a global symmetry of the system. The configuration space dynamics can be described, in this sense, only by the constraints, i.e. without independent equations of motion. In this context, the BRST variation $Q\eta_a$ in the sector of variables of ghost number -1 can be identified with the equations of motion, which may be not necessarily lagrangian, as it was subtly noticed in ref. [5].

The adjoint action of Q gives rise to a nilpotent differentiation $D : \Lambda^n_m(\mathcal{M}) \to \Lambda^n_{m+1}(\mathcal{M})$:

$$DA = [Q,A]\,, \qquad \forall\, A \in \Lambda(\mathcal{M})\,, \tag{3.26}$$

$$D^2 = 0\,, \qquad \mathrm{gh}(D) = 1\,. \tag{3.27}$$

The corresponding cohomology group can be decomposed $\mathcal{H}(D) = \bigoplus \mathcal{H}^n_m(D)$ according to the bi-grading by the p-degree and the ghost number.

The weak Poisson structure enters the expansion (3.21) through the bivector Π. The latter is a D-cocycle as well as the vector Γ. The third relation in (3.23) means that Π is a Poisson bivector up to a D-coboundary. Similarly, the second relation in (3.24) characterizes Γ as weak Poisson vector field on $\mathcal{M}$.

The higher polyvectors entering S and H contribute to various compatibility conditions between Q, Π and Γ, which result from the Jacobi identity for the Schouten brackets and the triviality of the D-cohomology groups $\mathcal{H}^n_m(D)$ with $m < n$. The last fact can be seen as follows: According to 1, $\mathrm{Deg}(A) > \mathrm{gh}(A)$ implies $\deg(A) > 0$, but D is the perturbation of δ by terms of higher r-degree, and the triviality of the D-cocycle A follows immediately from acyclicity of δ in strictly positive r-degree (3.14).

Let us show that the space of physical observables $C^\infty(N) = C^\infty(M)^{\mathrm{pr}}/C^\infty(M)^{\mathrm{triv}}$ is isomorphic to $\mathcal{H}^0_0(D)$. Substituting a general expansion

$$\Lambda^0_0(\mathcal{M}) \ni F = \sum_{n=0}^{\infty} F_n(\phi) = f(x) + \eta_a V^a_\alpha(x) c^\alpha + \cdots\,, \qquad \deg(F_n) = n\,, \tag{3.28}$$

into the D-closedness condition

$$DF = [Q, F] = (S, F)|_{\phi^*=0} = 0\,, \tag{3.29}$$

we get a sequence of equations of the form

$$\delta F_{n+1} = B_n(F_0, \ldots, F_n)\,, \qquad \deg(B_n) = n\,, \tag{3.30}$$

where the nilpotent differential (3.12) is now truncated to $\delta = T_a \partial/\partial \eta_a$, which is the usual Koszul-Tate differential associated to the constraint surface Σ. One may check that the first of the equations (3.30) is nothing but the projectibility condition for the function $f(x)$:

$$[R_\alpha, f] = V^a_\alpha T_a\,. \tag{3.31}$$

Using the identity $D^2 F \equiv 0$, one can see that the r.h.s. of the n-th equation is δ-closed provided all the previous equations are satisfied. Since $\mathcal{H}_n(\delta) = 0$ for all $n > 0$, we conclude that (i) any projectible function $f \in C^\infty(M)^{\mathrm{pr}}$ is lifted to a D-cocycle $F \in C^\infty(\mathcal{M})$ and (ii) any two equivalent (in the sense of (2.8)) functions $f_1, f_2 \in C^\infty(M)^{\mathrm{pr}}$ belong to the same cohomology class upon the lift, i.e.

$$F_1 - F_2 = DK\,, \qquad K = K^a \eta_a + \cdots\,. \tag{3.32}$$

This establishes the isomorphism $C^\infty(N) \simeq \mathcal{H}^0_0(D)$.

Now consider the following brackets on $C^\infty(\mathcal{M})$:

$$\{A, B\} = (A, (S, B))|_{\phi^*=0} = [A, [\Pi, B]]\,. \tag{3.33}$$

From the third relation in (3.23) it follows that for any $A, B, C \in C^\infty(\mathcal{M})$ we have

$$\begin{aligned}(-1)^{\epsilon(A)\epsilon(C)}\{\{A, B\}, C\} &+ \mathrm{cycle}(A, B, C) = \\ &= D\Psi(dA, dB, dC) + \Psi(dDA, dB, dC) + (-1)^{\epsilon(A)}\Psi(dA, dDB, dC) + \\ &\quad + (-1)^{\epsilon(A)+\epsilon(B)}\Psi(dA, dB, dDC)\,.\end{aligned} \tag{3.34}$$

In other words, the brackets (3.33) induce a Poisson structure on the cohomology group $\mathcal{H}^0(D)$ considered as a supercommutative algebra. In particular, the physical observables form a closed Poisson subalgebra $\mathcal{H}^0_0(D)$.

Finally, the evolution of a physical observable $[A] \in \mathcal{H}^0_0(D)$ represented by a D-cocycle A is determined by the equation

$$\dot{A} = [\Gamma, A] = (H, A)|_{\phi^*=0} \,. \tag{3.35}$$

In view of rels. (3.24) it defines a one-parameter automorphism of the Poisson algebra $\mathcal{H}^0_0(D)$.

We may now summarize that the spaces of physical observables $C^\infty(N)$, physical evolutions $\Lambda^1(N)$ and the Poisson structures on N are in one-to-one correspondence respectively with the cohomology groups $\mathcal{H}^0_0(D)$, $\mathcal{H}^1_0(D)$ and $\mathcal{H}^2_0(D)$ of the BRST operator D.

4. Quantization by means of formality theorem

The aim of this section is to perform the deformation quantization of the generic gauge system endowed with the weak Poisson brackets (3.33). For this purpose we apply the covariant version [13, 14] of Kontsevich's formality theorem [15], which states the existence of quasi-isomorphism

$$F : \Lambda(\mathcal{M}) \rightsquigarrow D(\mathcal{M}) \tag{4.1}$$

between two differential graded Lie algebras:[4] the algebra $(\Lambda(\mathcal{M}), [\,\cdot\,,\,\cdot\,], 0)$ of polyvector fields equipped with the Schouten bracket and trivial differential, and the algebra $(D(\mathcal{M}), [\,\cdot\,,\cdot\,]_{\mathcal{G}}, \delta)$ of polydifferential operators $D(\mathcal{M}) = \bigoplus D^k(\mathcal{M})$ on $\mathcal{M}$, where $D^k(\mathcal{M})$ is the space of differential operators acting on k functions, $[\,\cdot\,,\,\cdot\,]_{\mathcal{G}}$ is the Gerstenhaber bracket and $\delta : D^k(\mathcal{M}) \to D^{k+1}(\mathcal{M})$ is the Hochschild differential. Notice that if $\Lambda(\mathcal{M})$ and $D(\mathcal{M})$ are regarded as differential graded Lie algebras, the multiplicative gradings are shifted by -1.

To any collection of polyvector fields $A_1, \ldots, A_n$ of degrees $k_1, \ldots, k_n$ the map F assigns a polydifferential operator $F_n(A_1, \ldots, A_n) \in D^m(\mathcal{M})$ acting on $m = 2 - 2n + \sum_{i=1}^n k_i$ functions such that the following semi-infinite sequence of quadratic relations is satisfied ($n \geq 1$):

$$\begin{aligned} \delta F_n(A_1, \ldots, A_n) = \\ = \frac{1}{2} \sum_{k,l\geq 1,\, k+l=n} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm [F_k(A_{\sigma(1)}, \ldots, A_{\sigma(k)}), F_l(A_{\sigma(k+1)}, \ldots, A_{\sigma(k+l)})]_{\mathcal{G}} + \\ + \sum_{i \neq j} \pm F_{n-1}([A_i, A_j], A_1 \ldots, \hat{A}_i, \ldots, \hat{A}_j, \ldots, A_n) \,. \end{aligned} \tag{4.2}$$

Here S_n is the group of permutations of n letters and the caret denotes omission. The rule for determining signs in the above formula is rather intricate. For example, the sign factor in the first sum depends on the permutation σ, on degrees of A_i and on the numbers k and l.

[4] More precisely, here we mean the generalization of the formality theorem to the case of supermanifolds. No such a generalization has been published yet, although its existence was never doubted, as far as we could know.

Let us also mention the symmetry property

$$F_n(A_1,\ldots,A_i,\ldots,A_j,\ldots,A_n) = (-1)^{\epsilon(A_i)\epsilon(A_j)} F_n(A_1,\ldots,A_j,\ldots,A_i,\ldots,A_n)\,, \tag{4.3}$$

and the "boundary" condition

$$F_1(A)(f_1,\ldots,f_m) = (-1)^m A(df_1,\ldots,df_m)\,, \qquad \forall A \in \Lambda^m(\mathcal{M})\,. \tag{4.4}$$

Let us now apply the identities (4.2) to $F_n(S,\ldots,S)$. In view of the master equation

$$[S,S] = 0\,, \tag{4.5}$$

the sum in the second line of (4.2) vanishes and the whole sequence of relations can be combined in a single equation

$$[\hat{S},\hat{S}]_{\mathcal{G}} = 0 \tag{4.6}$$

for the inhomogeneous polydifferential operator

$$D(\mathcal{M})[[\hbar]] \ni \hat{S} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} F_n(S,\ldots,S)\,, \tag{4.7}$$

$\hbar$ being formal parameter. Here we use the fact that the Hochcshild differential δ is an inner derivation of the Gerstenhaber algebra, namely,

$$\delta = [F_0,\,\cdot\,]_{\mathcal{G}}\,, \tag{4.8}$$

where $F_0 \in D^2(\mathcal{M})$ is the multiplication operator, $F_0(f,g) = fg$, $\forall f,g \in C^\infty(\mathcal{M})$. Expanding $\hat{S}$ in the sum of homogeneous polydifferential operators

$$\begin{aligned} \hat{S} &= \sum_{n=0}^{\infty} \hat{S}^n = \hat{A} + \hat{Q} + \hat{\Pi} + \hat{\Psi} + \hat{S}^4 + \cdots\,, \\ D^n(\mathcal{M})[[\hbar]] \ni \hat{S}^n &= \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \sum_{l_1+\cdots+l_k = n+2(k-1)} F_k(S^{l_1},\ldots,S^{l_k})\,. \end{aligned} \tag{4.9}$$

and taking into account (4.3) and (4.4), we can write

$$\begin{aligned} \hat{A} &= \frac{\hbar^2}{2} F_2(Q,Q) + \frac{\hbar^3}{2} F_3(Q,Q,\Pi) + O(\hbar^4)\,, & \hat{Q}(f) &= -\hbar Q(df) + O(\hbar^2)\,, \\ \hat{\Pi}(f,g) &= fg + \frac{\hbar}{2}\Pi(df,dg) + O(\hbar^2)\,, & \hat{\Psi}(f,g,h) &= -\hbar \Psi(df,dg,dh) + O(\hbar^2)\,, \end{aligned} \tag{4.10}$$

for $f,g,h \in C^\infty(\mathcal{M})$. Now substituting (4.9) into the "Gerstenhaber master equation" (4.6), we get a sequence of relations, the first four of which are

$$\begin{aligned} &i)\ [\hat{A},\hat{Q}]_{\mathcal{G}} = 0\,, && iii)\ [\hat{Q},\hat{\Pi}]_{\mathcal{G}} = -[\hat{A},\hat{\Psi}]_{\mathcal{G}}\,, \\ &ii)\ [\hat{Q},\hat{Q}]_{\mathcal{G}} = -2[\hat{A},\hat{\Pi}]_{\mathcal{G}}\,, && iv)\ [\hat{\Pi},\hat{\Pi}]_{\mathcal{G}} = -2[\hat{Q},\hat{\Psi}]_{\mathcal{G}} - 2[\hat{A},\hat{S}^4]_{\mathcal{G}}\,. \end{aligned} \tag{4.11}$$

Let us suppose $\hat{A} = 0$, for a while. Then rels. (4.11) can be interpreted as follows:

(ii) The homological vector field Q is lifted to the formal differential operator

$$\hat{Q} : C^{\infty}(\mathcal{M})[[\hbar]] \to C^{\infty}(\mathcal{M})[[\hbar]] \,.$$

The latter is odd and nilpotent,

$$\hat{Q}^2 = 0 \,. \tag{4.12}$$

Let us denote by $\mathcal{H}(\hat{Q}) = \bigoplus \mathcal{H}^n(\hat{Q}) = \ker \hat{Q}/\mathrm{im}\hat{Q}$ the corresponding cohomology group, graded naturally by the ghost number.

(iv) The bidifferential operator

$$\hat{\Pi} : C^{\infty}(\mathcal{M})[[\hbar]] \otimes C^{\infty}(\mathcal{M})[[\hbar]] \to C^{\infty}(\mathcal{M})[[\hbar]]$$

defines the $*$-product

$$f * g = \hat{\Pi}(f,g) = fg + \frac{\hbar}{2}\{f,g\} + O(\hbar^2) \,, \tag{4.13}$$

which is a formal deformation of the pointwise product of functions in the "direction" of the weak Poisson bracket (3.33). This $*$-product possesses the property of weak associativity:

$$\begin{aligned}(f * g) * h - f * (g * h) \,=\, & \hat{Q}(\hat{\Psi}(f,g,h)) + \hat{\Psi}(\hat{Q}(f),g,h) + (-1)^{\epsilon(f)}\hat{\Psi}(f,\hat{Q}(g),h) + \\ & + (-1)^{\epsilon(f)+\epsilon(g)}\hat{\Psi}(f,g,\hat{Q}(h)) \,,\end{aligned} \tag{4.14}$$

for all $f,g,h \in C^{\infty}(\mathcal{M})[[\hbar]]$.

(iii) $\hat{Q}$ differentiates the $*$-product:

$$\hat{Q}(f * g) = \hat{Q}(f) * g + (-1)^{\epsilon(f)} f * \hat{Q}(g) \,. \tag{4.15}$$

Together, rels. (4.14), (4.15) suggest that the weakly associative $*$-product on $C^{\infty}(\mathcal{M})[[\hbar]]$ induces an *associative* $*$-product on the cohomology group $\mathcal{H}(\hat{Q})$. In particular, $\mathcal{H}^0(\hat{Q})$ - the group of the $\hat{Q}$-cohomology of ghost number zero — inherits the structure of $*$-subalgebra in $H(\hat{Q})$. It is this subalgebra which is identified with the algebra of *quantum observables.*

When $\hat{A} \neq 0$, the above interpretation of rels. (ii), (iii), (iv) becomes incorrect: The nonzero value of $\hat{A}$ may break the nilpotency of the operator $\hat{Q}$ as well as the property of the weak associativity (4.14) and the Liebniz rule (4.15). One may thus regard the function $\hat{A}$, having no classical counterpart, as a *quantum anomaly.* By definition $\mathrm{gh}(\hat{A}) = 2$, and hence it is expanded as

$$\hat{A} = c^{\alpha}c^{\beta}f_{\alpha\beta}(x,\hbar) + c^{\alpha}c^{\beta}c^{\gamma}f^{a}_{\alpha\beta\gamma}(x,\hbar)\eta_a + \cdots \,. \tag{4.16}$$

Actually, the function $F_2(Q,Q)$ vanishes identically [15], so the quantum anomaly starts with $\hbar^3$. The last fact agrees well with the absence of one-loop anomalies for the Weyl symbols of operators.

Let us now expand $\hat{A}$ in powers of $\hbar$:

$$\hat{A} = \hbar^n A^{(n)} + \hbar^{n+1}A^{(n+1)} + \cdots \,, \qquad n \geq 3 \,. \tag{4.17}$$

If the leading term $A^{(n)}$ is not equal to zero identically, we refer to n as the order of the anomaly $\hat{A}$. From the first relation in (4.11) follows that $A^{(n)}$ is a D-cocycle:

$$DA^{(n)} \equiv \left[Q, A^{(n)}\right] = 0 \;\Rightarrow\; \left[A^{(n)}\right] \in \mathcal{H}^0_2(D)\,. \tag{4.18}$$

The anomaly $\hat{A}$ is said to be trivial at the lowest order n, if $[A^{(n)}] = 0$. In this case, there exists a function B such that $A^{(n)} = DB$, and one can remove the leading term $A^{(n)}$ of the anomaly by an appropriate canonical transformation. Namely, applying (3.17) to S with $G = \hbar^{n-1}B$, we get a new solution $S' = S - \hbar^{n-1}A^{(n)} + \cdots$ to the classical master equation (4.5) and the corresponding solution $\hat{S}'$ to the Gerstenhaber master equation (4.6); but the order of the anomaly term $\hat{A}'$ is now equal to or grater than $n+1$.

If $A'^{(n+1)}$ happens to be a D-coboundary again, we may repeat the above procedure once again and get an anomaly of order $\geq n+2$. Doing this inductively on n, we can shift the anomaly to higher orders in $\hbar$ until we face with a nontrivial D-cocycle. In this case the procedure stops and we get a genuine quantum anomaly, which cannot be removed by the canonical transform (3.17). There is, however, a hypothetical possibility to cancel this anomaly out by means of a nontrivial deformation of S (i.e. a deformation which cannot be induced by a canonical transform):

$$S \to S(\hbar) = S + \hbar S_1 + \cdots\,, \qquad [S(\hbar), S(\hbar)] = 0\,. \tag{4.19}$$

The existence of such a deformation and a concrete recipe for its construction require a separate study going beyond the scope of the present paper.

Now let us formulate a quantum counterpart for the classical equation of motion (3.35). In order to do this consider the inhomogeneous polyvector $W = H + S$. The classical master equations (3.4) are obviously equivalent to the following one:

$$[W, W] = 0\,. \tag{4.20}$$

Proceeding analogously to the previous case with S, we apply the formality map to W and get inhomogeneous polydifferential operator $\hat{W} \in D(\mathcal{M})[[\hbar]]$ satisfying the Gerstenhaber master equation

$$[\hat{W}, \hat{W}]_{\mathcal{G}} = 0\,. \tag{4.21}$$

Since $\epsilon(H) = 1$, $\hat{W}$ is at most linear in H, as is clear from (4.3). Thus $\hat{W} = \hat{H} + \hat{S}$, where

$$\hat{H} = \sum_{n=0}^{\infty} \frac{\hbar^{n+1}}{n!} F_{n+1}(H, S, \ldots, S) \tag{4.22}$$

Expanding the formal polydifferential operator $\hat{H}$ in its homogeneous components, we get

$$\hat{H} = \sum_{n=0}^{\infty} H^n = \hat{A}' + \hat{\Gamma} + \hat{\Xi} + \cdots\,, \tag{4.23}$$

where

$$\begin{aligned} \hat{A}' &= \hbar^2 F_2(Q, \Gamma) + \hbar^3 F_3(Q, \Gamma, \Pi) + O(\hbar^4)\,, \\ \hat{H}(f) &= -\hbar\Gamma(df) + O(\hbar^2)\,, \\ \hat{\Xi}(f, g) &= \hbar\Xi(df, dg) + O(\hbar^2)\,, \end{aligned} \tag{4.24}$$

and $f, g \in C^\infty(\mathcal{M})$. Let us suppose that the anomaly $\hat{A}$ equals to zero. Then substituting (4.23) into (4.21), we get

$$\begin{aligned} &i) \quad [\hat{Q}, \hat{A}']_{\mathcal{G}} = 0\,, \\ &ii) \quad [\hat{Q}, \hat{\Gamma}]_{\mathcal{G}} = -[\hat{A}', \hat{\Pi}]_{\mathcal{G}}\,, \\ &iii) \ [\hat{\Gamma}, \hat{\Pi}]_{\mathcal{G}} = [\hat{Q}, \hat{\Xi}]_{\mathcal{G}} + [\hat{\Psi}, \hat{A}']_{\mathcal{G}}\,. \end{aligned} \tag{4.25}$$

By definition, $\hat{A}'$ is a scalar function of ghost number 1. Consequently,

$$\hat{A}' = c^\alpha f_\alpha(x, \hbar) + c^\alpha c^\beta f^a_{\alpha\beta}\eta_a + \cdots\,. \tag{4.26}$$

Having no classical counterpart, it represents one more quantum anomaly related to the dynamics of the constrained gauge system. Again, $F_2(Q, \Gamma) \equiv 0$, and the expansion (4.24) for $\hat{A}'$ starts actually with $\hbar^3$:

$$\hat{A}' = \hbar^n A'^{(n)} + \hbar^{n+1} A'^{(n+1)} + \cdots\,, \qquad n \geq 3\,. \tag{4.27}$$

¿From the first relation of (4.25) it then follows that the leading term $A'^{(n)}$ is a D-cocycle, i.e. $[A'^{(n)}] \in \mathcal{H}^0_1(D)$. Again, if $A'^{(n)} = DB'$ one can remove this term by an appropriate canonical transform in perfect analogy to the case of $\hat{A}$.

If now $\hat{A}' = 0$, then the last two relations of (4.25) suggest that the operator

$$\hat{\Gamma} : C^\infty(\mathcal{M})[[\hbar]] \to C^\infty(\mathcal{M})[[\hbar]]$$

commutes with $\hat{Q}$ and differentiates the $*$-product up to homotopy,

$$\hat{\Gamma}(f * g) - (\hat{\Gamma} f) * g - f * (\hat{\Gamma} g) = \hat{Q}(\hat{\Xi}(f, g)) + \hat{\Xi}(\hat{Q}(f), g) + (-1)^{\epsilon(f)}\hat{\Xi}(f, \hat{Q}(g))\,. \tag{4.28}$$

As a consequence, $\hat{\Gamma}$ induces a differentiation of the algebra of physical observables $\mathcal{H}^0_0(\hat{Q})$.

Now if both the anomalies $\hat{A}$ and $\hat{A}'$ vanish, the quantum evolution is governed by the equation

$$\dot{O} = \hat{\Gamma} O\,, \tag{4.29}$$

where O is a D-cocycle representing an observable $[O] \in \mathcal{H}^0_0(\hat{Q})$.

In field theory, one should be cautious about the aforementioned possibilities of removing quantum anomalies because of possible divergences appearing when the polydifferential operators are applied to local functionals. Even when $\mathcal{H}^0_1(D) = \mathcal{H}^0_2(D) = 0$ the formally BRST-exact contributions can diverge resulting in anomalies, whereas after a regularization they may happen to be no longer BRST-closed.

5. Weak Lie algebroids and a topological sigma-model

The concept of constrained gauge algebra, exposed in section 2, admits a straightforward generalization to the case when the constraints T_a and the gauge algebra generators R_α are considered to be sections of some vector bundles over M rather than sets of functions and vector fields. Namely, we may set

$$T = T_a e^a \in \Gamma(E_1)\,, \qquad R = R^i_\alpha e^\alpha \otimes \partial_i \in \Gamma(E_2 \otimes TM)\,,$$

where $\{e^a\}$ and $\{e^\alpha\}$ are local frames of two vector bundles $E_1 \to M$ and $E_2 \to M$, and $\partial_i = \partial/\partial x^i$ is the natural frame in the tangent bundle of the base M.

Notice that all the structure relations (2.1)–(2.6) of the constrained gauge algebra as well as the projectibility conditions (2.9) are form-invariant under the bundle automorphisms. The shell Σ is now identified with the zero locus of the section T; the latter is supposed to intersect the base M transversally in order for $\Sigma \subset M$ to be a smooth submanifold (cf. (2.2)). The section R defines (and is defined by) a bundle homomorphism $R : E_2^* \to TM$. The regularity condition (2.5) is then equivalent to the injectivity of $R|_\Sigma$. Rel. (2.4) suggests that $\mathrm{im}(R|_\Sigma) \subset T\Sigma \subset TM|_\Sigma$, whereas rel. (2.6) identifies $\mathrm{im}(R|_\Sigma)$ as an integrable distribution on Σ. In other words, $E_2|_\Sigma$ is just an injective Lie algebroid over Σ with the anchor $\bar{R} = R|_\Sigma$. The constructions of section 2 correspond to the special case where both E_1 and E_2 are trivial vector bundles.

In view of an apparent similarity to the definition of the Lie algebroid, it is natural to call the quadruple (E_1, E_2, R, T) a *weak Lie algebroid*, as the integrability of the (anchor) distribution R takes place only on the (constraint) surface $\Sigma = \{x \in M | T(x) = 0\}$.

Now let us describe the BRST-imbedding of a gauge system involving a weak Lie algebroid (E_1, E_2, R, T) and a projectible vector field $V \in \Lambda^1(M)^{\mathrm{pr}}$ respecting a weak Poisson structure $P \in \Lambda^2(M)^{\mathrm{pr}}$. In this case, the supermanifold of fields $\mathcal{M}$ is chosen to be the direct sum $\Pi E_1 \oplus \Pi E_2$ of odd vector bundles over M. Let η_a and c^α denote linear coordinates in fibers of ΠE_1 and ΠE_2 over a trivializing chart $U \subset M$ with local coordinates x^i. The whole field-antifield supermanifold $\mathcal{N}$ is associated with the total space of the vector bundle (cf. (3.1))

$$E_1^* \oplus E_2^* \oplus \Pi E_1 \oplus \Pi E_2 \oplus \Pi T^* M\,. \tag{5.1}$$

Choosing a linear connection $\nabla = \nabla_1 \oplus \nabla_2$ on $E_1 \oplus E_2$, one can endow $\mathcal{N}$ with an exact antisymplectic structure $\Omega = d\Theta$, where

$$\begin{aligned} \Theta &= (\overset{*}{x}_i dx^i + \overset{*}{\eta}{}^a \nabla_1 \eta_a + \overset{*}{c}_\alpha \nabla_2 c^\alpha)\,, \\ \nabla_1 \eta_a &= d\eta_a + dx^i \Gamma^b_{ia}(x)\eta_b\,, \qquad \nabla_2 c^\alpha = dc^\alpha + dx^i \Gamma^\alpha_{i\beta}(x) c^\beta\,. \end{aligned} \tag{5.2}$$

The corresponding antibrackets of fields $\phi^A = (x^i, c^\alpha, \eta_a)$ and antifields $\phi^*_A = (\overset{*}{x}_i, \overset{*}{c}_\alpha, \overset{*}{\eta}{}^a)$ read

$$\begin{aligned} &(\overset{*}{c}_\alpha, c^\beta) = \delta^\beta_\alpha\,, \quad (\overset{*}{x}_i, c^\alpha) = \Gamma^\alpha_{i\beta} c^\beta\,, \qquad (\overset{*}{x}_i, \overset{*}{c}_\alpha) = -\Gamma^\beta_{i\alpha}\, \overset{*}{c}_\beta\,, \\ &(\overset{*}{\eta}{}^a, \eta_b) = \delta^a_b\,, \quad (\overset{*}{x}_i, \eta_a) = \Gamma^a_{ib}\eta_b\,, \qquad (\overset{*}{x}_i, \overset{*}{\eta}{}^a) = -\Gamma^a_{ib}\, \overset{*}{\eta}{}^b\,, \\ &(\overset{*}{x}_i, x^j) = \delta^j_i\,, \quad (\overset{*}{x}_i, \overset{*}{x}_j) = R^\alpha_{ij\beta}\, \overset{*}{c}_\alpha c^\beta + R^b_{ij\,a}\, \overset{*}{\eta}{}^a \eta_b\,, \end{aligned} \tag{5.3}$$

and the other brackets vanish. Here $R(x)^\alpha_{ij\beta}$ and $R(x)^b_{ij\,a}$ are the curvatures of ∇_1 and ∇_2.

As with the case of trivial vector bundles E_1 and E_2, considered in section 3, all the ingredients of the gauge system are combined into the pair of functions S and H subject to the master equations (3.4). The covariance of the antibrackets (5.3) under the bundle automorphisms suggests the coefficients of the expansions (3.8), (3.9) for S and H to transform homogeneously i.e. as the tensors associated with the vector bundle (5.1).

Since the antibrackets (5.3) have not the canonical form, we cannot apply the formality theorem directly to perform the deformation quantization of the gauge system as it has been done in section 4. The generalization of the formality theorem to arbitrary antisymplectic manifolds (which are not odd cotangent bundles with the canonical antibrackets) still remains an interesting open question. For lack of a rigorous mathematical treatment we will use a less rigor but more descriptive approach of topological sigma-models of ref. [16], where it was originally used to elucidate the Kontsevich quantization formula for Poisson manifolds. Below, we briefly explain how this approach works in a more general situation of the weak Poisson structure.

Following [8, 16, 17], we consider a topological sigma-model having $\mathcal{N}$ as target space, and whose world sheet is given by the supermanifold ΠTU, where U is a closed two-dimensional disk. Let u^μ, $\mu = 1, 2$, denote local coordinates on U and θ^μ denote odd coordinates on the fibers of ΠTU. The natural volume element on ΠTU is given by $d\mu = d^2u d^2\theta$. We choose the odd coordinates θ^μ to have ghost number 1.

Let $\Phi^I = (\phi^A, \phi^*_B)$ be a collective notation for all the fields and antifields. Each field configuration $\Phi^I(u, \theta)$ defines an imbedding of ΠTU into $\mathcal{N}$. Expanding superfield Φ^I in θ's one gets a collection of ordinary fields on U,

$$\Phi^I(u,\theta) = \Phi^I_0(u) + \theta^\mu\Phi^I_\mu(u) + \theta^\mu\theta^\nu\Phi^I_{\mu\nu}(u)\,, \tag{5.4}$$

The parities and ghost numbers of the component fields $\Phi^I_0, \Phi^I_\mu, \Phi^I_{\mu\nu}$ are uniquely determined by those of Φ^I and θ^μ. Each superfield Φ^I is naturally identified with an inhomogeneous differential form on U; in doing so, the role of exterior differential is played by the supercovariant derivative

$$\mathrm{D} = \theta^\mu\frac{\partial}{\partial u^\mu}\,, \quad \Rightarrow \quad \mathrm{D}^2 = 0\,. \tag{5.5}$$

With any solution $S(\Phi)$ of the master equation $(S, S) = 0$ one can associate the sigma-model [17, 8], whose master action reads

$$\mathcal{A}[\Phi] = \int_{\Pi TU} d\mu\left(\Theta_I(\Phi)\mathrm{D}\Phi^I - S(\Phi)\right)\,, \qquad \mathrm{gh}(\mathcal{A}) = 0\,. \tag{5.6}$$

The antibrackets (5.3) on the target space $\mathcal{N}$ give rise to antibrackets on the configuration space of superfields:

$$(F,G)' = \int_{\Pi TU} d\mu\left(\frac{F\overleftarrow{\delta}}{\delta\Phi^I(u,\theta)}\Omega^{IJ}\frac{\overrightarrow{\delta}G}{\delta\Phi^J(u,\theta)}\right)\,, \qquad \Omega_{IJ}\Omega^{JK} = \delta^K_I\,, \tag{5.7}$$

for any functionals $F[\Phi]$ and $G[\Phi]$. By construction, the action (5.6) obeys the classical master equation

$$(\mathcal{A},\mathcal{A})' = 2\int_{\Pi TU} d\mu \mathrm{D}[\Theta(\Phi)_I\mathrm{D}\Phi^I + S(\Phi)] + \int_{\Pi TU} d\mu(S,S)(\Phi) = 0\,. \tag{5.8}$$

The second term vanishes due to the classical master equation $(S, S) = 0$, while vanishing of the first term is due to appropriate boundary conditions on the component fields (5.4). In particular, it is assumed that

$$\phi^*_A|_{\partial U} = 0\,. \tag{5.9}$$

The BV quantization of the model (5.6) requires an integration measure (or density) to be introduced on the configuration space of fields $\Phi^I(u,\theta)$. As well as the antibrackets, the measure can be obtained from a normal, nowhere vanishing density on the target space $\mathcal{N}$. Recall that a density ρ is said to be *normal* if there is an atlas of Darboux's coordinates (3.3) on $\mathcal{N}$ in which $\rho = 1$. Any normal density gives rise to the nilpotent odd Laplace operator $\Delta : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{N})$:

$$\Delta f = \mathrm{div}_\rho X_f\,, \qquad \Delta^2 = 0\,, \tag{5.10}$$

$X_f = (f, \cdot\,)$ being the hamiltonian vector field corresponding to f.

Given a normal density ρ on $\mathcal{N}$, the natural integration measure on the configuration space of fields can be chosen as

$$\mathcal{D}\Phi = \rho'[\Phi] \prod_{z\in\Pi TU} \delta\Phi^1(z)\cdots\delta\Phi^N(z)\,, \qquad \rho'[\Phi] = \prod_{z\in\Pi TU} \rho(\Phi(z))\,, \tag{5.11}$$

where $N = \dim\mathcal{N}$. The functional counterpart of the odd Laplacian (5.10) is then given by

$$\Delta' = \int_{\Pi TU} d\mu\, \rho'^{-1} \frac{\delta}{\delta\Phi^I(u,\theta)} \rho' \Omega^{IJ} \frac{\delta}{\delta\Phi^J(u,\theta)}\,. \tag{5.12}$$

Although, in the field-theoretical context, the odd Laplacian is known to be an ill-defined operator, we claim that after an appropriate regularization

$$\Delta'\mathcal{A} = 0\,. \tag{5.13}$$

The reason is as follows. Since Δ' is a local operator, one can compute its action passing to suitable Darboux's coordinates on $\mathcal{N}$ in which $\rho' = 1$. In these coordinates, Δ' becomes a homogeneous second order operator. Consequently,

$$\Delta'\mathcal{A} = \delta(0)(\text{something})\,, \tag{5.14}$$

where the δ-function on ΠTU is defined as $\delta(z) = \theta^1\theta^2\delta(u^1)\delta(u^2)$. So, the right hand side of (5.14) vanishes due to the θ-part of the δ-function.

As a consequence the classical master action $\mathcal{A}$ obeys also the quantum master equation

$$\frac{1}{2}(\mathcal{A},\mathcal{A})' = \hbar\Delta'\mathcal{A}\,. \tag{5.15}$$

By definition, the BRST observables are functionals $\mathcal{O}[\Phi]$ annihilated by the quantum BRST operator Ω:

$$\Omega\mathcal{O} \equiv -i\hbar\Delta'\mathcal{O} + (\mathcal{A},\mathcal{O})' = 0\,. \tag{5.16}$$

Using eq. (5.15) and the fact that Δ' is a derivation of the antibrackets (5.7) of degree 1, one can see that $\Omega^2 = 0$. A particular class of BRST observables is provided by the D-closed functions from $C^\infty(\mathcal{M})$ of ghost number zero. (Recall that the cohomology classes of these

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cocycles are precisely identified with the classical observables of our gauge system.) Let us set $\mathcal{O}_f(u) = f(\phi(u,0))$ for $u \in \partial U$. From the definition (3.29) of a D-cocycle and the boundary condition (5.9) for antifields it then follows that

$$(\mathcal{A}, \mathcal{O}_f(u))' = (S, f)(\phi(u,0)) = (Q, f)(\phi(u,0)) = 0\,. \tag{5.17}$$

Furthermore, passing to suitable Darboux's coordinates one can also see that $\Delta'\mathcal{O}_f(u) \sim \delta(0)$ and hence, after an appropriate regularization, $\Delta'\mathcal{O}_f(u) = 0$. So, any classical observable gives rise to the BRST observable.

By analogy with [16], we define the weakly associative $*$-product of two classical observables $f, g \in C^\infty(\mathcal{M})$ by the following path integral:

$$(f * g)(x) = \int_{\mathcal{L}} \mathcal{D}\Phi \mathcal{O}_f(u)\mathcal{O}_g(v)\delta[\phi(w) - x]e^{\frac{i}{\hbar}\mathcal{A}[\Phi]}\,. \tag{5.18}$$

Here u, v, w are three distinct points on the boundary circle ∂U, and the integral is taken over a suitable lagrangian submanifold $\mathcal{L} \subset \mathcal{N}$. Being a BRST observable, the integrand (5.18) is annihilated by the odd Laplacian (5.12) and the weak associativity of the $*$-product follows formally from the usual factorization arguments (see e.g. [16]).

Non-rigor though the path-integral arguments are, they will hopefully be found useful as a practical tool and as a starting point for constructing a more rigor deformation quantization of generic (not necessarily hamiltonian) gauge systems whose BRST embedding is proposed in this paper.

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